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The refraction of head seas by a long ship. Part 2. Waves of long wavelength

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It is known that head seas cannot travel without deformation along a cylinder of full constant cross-section, and recent calculations have indicated that the wave amplitude near the cylinder ultimately decreases as the waves travel along the cylinder, i.e. that the waves are refracted away from the axis of the cylinder. It was assumed in these calculations that the cross-section was a half-immersed circle of radius a of the same order as the wavelength $2\pi/K$, but the method can probably be adapted to arbitrary full constant cross-sections. (There is however another calculation which indicates that for a *thin ship* the wave amplitude ultimately increases.) In the present paper these calculations are extended. The circular section is again studied but it is now supposed that the wavenumber Ka may be small. Uniformly valid expressions for the wave potential are obtained which show that for small Ka the refraction becomes significant only when Kx (the dimensionless distance along the cylinder) is so large that the product $(Kx)^{\frac{1}{2}} v_0(Ka)$ is also large; here the function $v_0(Ka) \sim 2Ka$ arises in the solution of a certain eigenvalue problem. (The uniformly valid expressions also suggest an interpretation of the thin-ship calculation which resolves the apparent inconsistency.) The same method is applied to the waves generated by a pulsating source on an infinite cylinder, and similar results are obtained.

1. Introduction

In earlier work the effect of a long cylindrical ship on head seas was considered. Thus it was shown (Ursell 1968, hereafter referred to as II) that head seas cannot travel along such a ship without deformation. When the horizontal diameter $2a$ and the wavelength $2\pi/K$ are of comparable magnitude this deformation can be shown to consist of a progressive refraction away from the axis of the ship (Ursell 1975, hereafter referred to as IV); the total wave amplitude along the ship decreases like $(Kx)^{-\frac{1}{2}}$, where x is the distance measured along the ship. Another result relates to a ship with a thin wedge-like cross-section for which Ka is very small; it was shown in §5 of II that the amplitude of the diffracted wave increases like $(Kx)^{\frac{1}{2}}$ along the ship. It is of some interest to investigate how these results can be reconciled; also, in practical applications the parameter Ka is often quite small. In the present paper we shall accordingly be concerned with the asymptotic behaviour of the wave motion along the ship when Kx is large while Ka is small. The results are consistent with earlier results when Ka is of order unity.

The same methods will be used as in IV, except that the asymptotic treatment of IV will be replaced by a *uniformly asymptotic* treatment which remains valid when Ka is allowed to tend to zero. Repetition will be avoided where possible, and the results of IV will be freely quoted. As in IV, linearized theory will be used, the ship will be replaced by an infinitely long horizontal cylinder of constant cross-section on which the normal velocity is suitably prescribed, and the cross-section will be taken to be a half-immersed circle. Three problems were considered in IV. Problem 1 was concerned with head seas incident on a fixed semi-infinite ship, problem 2 with waves generated by a pulsating source on an infinite ship, and problem 3 with the wave pattern generated by an infinite nearly cylindrical ship moving along its length through still water. The mathematical treatments of problem 2 and problem 3 are almost identical, and problem 3 will therefore not be considered in the present paper. We shall find, as in IV, that the form of the asymptotic behaviour for large Kx depends only on the form of the singularities of the Fourier transform $\Phi(k, y, z)$ near $k = K$, and that this can be found without finding individual coefficients. For the sake of clarity it will be convenient to begin with problem 2. We shall then pass on to problem 1, which involves some additional complications.

2. Problem 2: a distributed pulsating source on an infinite cylinder

The same notation will be used as in §4 of IV. The velocity potential $\phi_2(x, y, z)\, e^{-i\sigma t}$ satisfies the equation of continuity

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi_2(x,y,z) = 0 \quad \text{in the fluid,} \tag{2.1}$$

the boundary condition

$$(K+\partial/\partial z)\,\phi_2 = 0 \quad \text{on the mean free surface} \quad z = 0, \quad r > a \tag{2.2}$$

and the boundary condition

$$\partial\phi_2/\partial r = v(x,\theta) \quad \text{on the cylinder} \quad r = a. \tag{2.3}$$

(The index 2 will usually be omitted in the calculations of the present section.) It is assumed, as in IV, that $v(x,\theta) = 0$ when $|x| > l$, where l is an arbitrary length, and that $v(x,\theta)$ is an even function of θ, but these restrictions can easily be removed. There is also the radiation condition that ϕ represents outward-travelling waves at infinity. A method of solving this problem was given in §4 of IV. Let $\Phi(k,y,z)$ and $V(k,\theta)$ be defined by the equations

$$\Phi(k,y,z) = \int_{-\infty}^{\infty} \phi(x,y,z)\, e^{-ikx}\, dx, \tag{2.4}$$

$$V(k,\theta) = \int_{-\infty}^{\infty} v(x,\theta)\, e^{-ikx}\, dx. \tag{2.5}$$

Then from equation (4.11) of IV we have (with $\alpha = i\beta$)

$$\Phi(k,y,z) = aV_0\,\frac{\pi\cos\beta\,\mathscr{T}(kr,\theta,\cos\beta)+\sin\beta\,\mathscr{R}(kr,\theta,\cos\beta)}{\pi\cos\beta\,T_0(k)+\sin\beta\,R_0(k)}$$
$$-aV_0\sum_{m=1}^{\infty}\frac{\pi\cos\beta\,T_{2m}(k)+\sin\beta\,R_{2m}(k)}{\pi\cos\beta\,T_0(k)+\sin\beta\,R_0(k)}\,\frac{\Psi_{2m}(kr,\theta,\cos\beta)}{K'_{2m}(ka)}, \tag{2.6}$$

where $K/k = \cos\beta$ $(0 < \beta < \frac{1}{2}\pi)$ and where the functions $\mathscr{T}$, $\mathscr{R}$ and Ψ_{2m} are defined in the appendix at the end of the present paper. They satisfy (2.1) and (2.2). The coefficients T_0, T_{2m} and R_0, R_{2m} are the coefficients in the expansions (over the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$)

$$T(k,\theta) = \frac{V(k,\theta)}{V_0} T_0(k) + \sum_{m=1}^{\infty} T_{2m}(k) \left\langle \frac{a}{K'_{2m}(ka)} \frac{\partial \Psi_{2m}}{\partial r} \right\rangle_{r=a} \tag{2.7}$$

and

$$R(k,\theta) = \frac{V(k,\theta)}{V_0} R_0(k) + \sum_{m=1}^{\infty} R_{2m}(k) \left\langle \frac{a}{K'_{2m}(ka)} \frac{\partial \Psi_{2m}}{\partial r} \right\rangle_{r=a}; \tag{2.8}$$

here $T(k,\theta) = \langle a\,\partial\mathscr{T}/\partial r\rangle$, $R(k,\theta) = \langle a\,\partial\mathscr{R}/\partial r\rangle$ and V_0 is an arbitrary normalizing constant. Angular brackets will be used to indicate that r is to be put equal to a after differentiation. The coefficients depend on $V(k,\theta)$; they are functions of k and involve K and a as parameters. The form (2.6) is appropriate when $k > K$; when $0 < k < K$ we write $\beta = i\alpha$ $(0 < \alpha < \infty)$ to obtain equation (4.11) of IV. This is the appropriate form since β was taken to be positive in (2.6) above, and since the contour in the k plane passes *below* $k = K$. [Note a misprint in equations (2.15) and (2.17) of IV: on the right-hand sides the symbol $\pm$ should be replaced by $\mp$. Note also that the left-hand side of equation (4.9) of IV should read $T(k,\theta)$.] Let us write

$$\sin\beta\,\Psi_0(kr,\theta,\beta) = \mathscr{S}(kr,\theta,\beta) = 2\pi\cos\beta\,\mathscr{T}(kr,\theta,\cos\beta) + 2\sin\beta\,\mathscr{R}(kr,\theta,\cos\beta), \tag{2.9}$$

$$S(k,\theta) = 2\pi\cos\beta\,T(k,\theta) + 2\sin\beta\,R(k,\theta), \tag{2.10}$$

$$S_{2m}(\beta,Ka) = 2\pi\cos\beta\,T_{2m}(k) + 2\sin\beta\,R_{2m}(k); \tag{2.11}$$

also let us write

$$N(kr,\theta,Ka) = N(Kr\sec\beta,\theta,\beta,Ka) = \mathscr{S}(kr,\theta,\beta) - \sum_{m=1}^{\infty} S_{2m}(\beta,Ka)\frac{\Psi_{2m}(kr,\theta,\cos\beta)}{K'_{2m}(ka)}. \tag{2.12}$$

Then we have

$$\Phi(k,y,z) = \frac{aV_0 N(Kr\sec\beta,\theta,\beta,Ka)}{S_0(\beta,Ka)}, \tag{2.13}$$

and

$$S(k,\theta) = \frac{V(k,\theta)}{V_0} S_0(\beta,Ka) + \sum_{m=1}^{\infty} S_{2m}(\beta,Ka) \left\langle \frac{a}{K'_{2m}(ka)} \frac{\partial \Psi_{2m}}{\partial r} \right\rangle. \tag{2.14}$$

We also have

$$\mathscr{S}(kr,\theta,-\beta) = 4\pi\cos\beta\,e^{-Kz}\cosh(Ky\tan\beta) - \mathscr{S}(kr,\theta,\beta). \tag{2.15}$$

3. Asymptotic evaluation for large Kx and small Ka

From (2.4) the potential

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(k,y,z)\,e^{ikx}\,dk \tag{3.1}$$

$$= \frac{1}{2\pi}\int \Phi(K\sec\beta,y,z)\exp(iKx\sec\beta)\frac{K\sin\beta}{\cos^2\beta}\,d\beta \tag{3.2}$$

may be determined in principle but not in practice, since $\Phi(k,y,z)$ has not been found here in an explicit form. We are however mainly concerned with the waves of wavenumber K for which Kx is large and which are near the cylinder, where y and z are bounded; in other words, we are concerned with contributions to the integral (3.2) from the saddle point $\beta = 0$; cf. p. 693 of IV. When Ka was not small or large it was found in IV that the denominator $S_0(\beta, Ka)$ does not vanish near $\beta = 0$, and that the asymptotic behaviour of the wave term $\phi_K(x, y, z)$ in $\phi(x, y, z)$ is then given by

$$\phi_K(x,y,z) \sim -\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi} \frac{KaV_0}{T_0(K)} e^{iKx} \Phi_*(K,y,z) \frac{1}{(Kx)^{\frac{1}{2}}} \tag{3.3}$$

through a straightforward expansion of $\Phi(k, y, z)$ about $k = K$. The potential $\Phi_*(K, y, z)$ is that solution of

$$(\partial^2/\partial y^2 + \partial^2/\partial z^2 - K^2)\,\Phi_* = 0 \tag{3.4}$$

which satisfies $\langle a\,\partial\Phi_*/\partial r\rangle = 0$ on $r = a$ and $\Phi_*(K, y, z) \sim -2\pi|Ky|\,e^{-Kz}$ as $|y| \to \infty$; see §2 of IV.

When Ka and $kr = Kr\sec\beta$ are both small, the convergent power-series expansions of the wave functions $\mathscr{T}$, $\mathscr{R}$ and Ψ_{2m} (given in the appendix) can be seen without great difficulty to be also asymptotic expansions. By retaining only a few terms (as in §4 of Ursell 1962, hereafter referred to as I) it is then possible to solve the systems (2.7) and (2.8) approximately for small Ka. A trial calculation for special velocity distributions of the form $v(x, \theta) = v_0(x)\cos\theta$ showed that the denominator

$$S_0(\beta, Ka) = 2\pi\cos\beta\, T_0(k) + 2\sin\beta R_0(k) \tag{3.5}$$

has a small real negative zero at $\beta = -\beta_0(Ka)$, say, where $\beta_0(Ka) \sim 2Ka$ as $Ka \to 0$. (It will be seen later, in §4, that $\beta_0(Ka)$ is independent of $v(x, \theta)$.) Equations (2.13) and (3.1) show that

$$\phi(x,y,z) = \frac{KaV_0}{2\pi} e^{iKx} \int \frac{N(Kr\sec\beta, \theta, \beta, Ka)}{S_0(\beta, Ka)} \exp\{iKx(\sec\beta - 1)\} \frac{\sin\beta}{\cos^2\beta} d\beta, \tag{3.6}$$

where $S_0(\beta, Ka)$ has a real simple zero $\beta = -\beta_0(Ka)$ near the saddle point $\beta = 0$, and where we are concerned with the contribution $\phi_K(x, y, z)$ from the neighbourhood of $\beta = 0$. Thus we must use a uniformly asymptotic technique (a simple modification of the method of steepest descents) which is applicable when the integrand has a simple pole near a saddle point (cf. appendix 2 of II, for the case of a double pole).

As in the ordinary method of steepest descents, take a new variable of integration v, as follows. Write $\frac{1}{2}v^2 = \sec\beta - 1 = 2\sin^2\frac{1}{2}\beta/\cos\beta$ and take the positive square root

$$v = 2\sin\tfrac{1}{2}\beta/(\cos\beta)^{\frac{1}{2}}; \tag{3.7}$$

then $v \sim \beta$ as $\beta \to 0$. Also write

$$v_0(Ka) = 2\sin\tfrac{1}{2}\beta_0/(\cos\beta_0)^{\frac{1}{2}}. \tag{3.8}$$

Near $\beta = 0$, let the non-oscillatory part of the integrand of (3.6) be expanded in a convergent series of the form

$$\frac{N(\quad)\sin\beta}{S_0(\beta, Ka)\cos^2\beta}\frac{d\beta}{dv} = \frac{A(Kr, \theta, Ka)}{v + v_0(Ka)} + \sum_{m=0}^{\infty} B_m(Kr, \theta, Ka)\,v^m, \tag{3.9}$$

where, from (3.15) below, we have $A(\quad) = -v_0 B_0(\quad)$. Thus the right-hand side of (3.9) is

$$\frac{v}{v+v_0} B_0(\quad) + \sum_1^{\infty} B_m(\quad) v^m.$$

Then, by the familiar argument used to prove Watson's lemma,

$$\int \frac{N(\quad)}{S_0(\beta, Ka)} \exp\{iKx(\sec\beta - 1)\} \frac{\sin\beta}{\cos^2\beta} \frac{d\beta}{dv} dv \tag{3.10}$$

$$\sim B_0(\quad) \int_L \frac{v \exp(\frac{1}{2}iKxv^2)}{v+v_0(Ka)} dv \tag{3.11}$$

$$+ \sum_{m=1}^{\infty} B_{2m}(\quad) \int_L v^{2m} \exp(\tfrac{1}{2}iKxv^2) dv, \tag{3.12}$$

where the path of integration in the complex v plane may be taken to be the path L from $v = -\infty \exp(\frac{1}{4}i\pi)$ through $v = 0$ to $v = \infty \exp(\frac{1}{4}i\pi)$. Retaining only the leading terms, we see that

$$\int \frac{N(\quad)}{S_0(\quad)} \exp(\quad) \frac{\sin\beta}{\cos^2\beta} d\beta \sim B_0(\quad) \left(\frac{2}{Kx}\right)^{\frac{1}{2}} F_1\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\} - B_2(\quad) \frac{(2\pi)^{\frac{1}{2}}}{(Kx)^{\frac{3}{2}}} e^{-\frac{1}{4}i\pi}, \tag{3.13}$$

where

$$F_1(\zeta) = \int_L \frac{w}{w+\zeta} \exp(iw^2) dw.$$

To find $A(\quad)$ multiply (3.9) by $v+v_0$ and let $\beta \to -\beta_0$ and $v \to -v_0$. Then

$$\begin{aligned} A(Kr, \theta, Ka) &= -N(Kr\sec\beta_0, \theta, -\beta_0, Ka) \frac{\sin\beta_0}{\cos^2\beta_0} \lim_{\beta\to-\beta_0} \left(\frac{v+v_0}{S_0(\beta, Ka)} \frac{d\beta}{dv}\right) \\ &= -N(\quad) \frac{\sin\beta_0}{\cos^2\beta_0} \frac{1}{(\partial S_0/\partial\beta)_{\beta=-\beta_0}}. \end{aligned} \tag{3.14}$$

To find $B_0(\quad)$, put $\beta = 0$ and $v = 0$ in (3.9). Then

$$0 = A(\quad)/v_0(Ka) + B_0(\quad),$$

whence

$$A(\quad) = -v_0(Ka) B_0(\quad). \tag{3.15}$$

An explicit expression can also be found for $B_2(\quad)$. For if it is assumed that

$$(Kx)^{\frac{1}{2}} v_0(Ka)$$

is large, then the asymptotic expansion for F_1 can be used (see appendix) and the calculation of equation (4.12) of IV then shows, after comparison with (3.3) above, that we must have

$$B_2(Kr, \theta, Ka) = \frac{1}{2\pi T_0(K)} \Phi_*(K, y, z) + \frac{B_0(Kr, \theta, Ka)}{v_0^2}, \tag{3.16}$$

where the two terms on the right become large and tend to cancel out when Ka tends to zero. From (3.6) and (3.13) we now find (restoring the index 2) that

$$\phi_{2K}(x,y,z) \sim \frac{KaV_0}{2^{\frac{1}{2}}\pi} e^{iKx} B_0^{(2)}(Kr,\theta,Ka)\,(Kx)^{-\frac{1}{2}} F_1\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\}$$
$$-\frac{KaV}{(2\pi)^{\frac{1}{2}}} e^{iKx} B_2^{(2)}(Kr,\theta,Ka)\frac{e^{-\frac{1}{4}i\pi}}{(Kx)^{\frac{3}{2}}}, \quad (3.17)$$

or, from (3.17),

$$\phi_{2K}(x,y,z) \sim -\frac{Kav_0^2}{2^{\frac{3}{2}}\pi^2}\frac{V_0}{T_0^{(2)}(K)}\frac{e^{iKx}}{(Kx)^{\frac{1}{2}}}\Phi_*(K,y,z)\,F_1\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\} \quad (3.18)$$

$$+\frac{2^{\frac{1}{2}}KaV_0}{\pi}\frac{e^{iKx}}{(Kx)^{\frac{3}{2}}} B_2(Kr,\theta,Ka)\,F_3\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\}, \quad (3.19)$$

where

$$F_3(\zeta) = \int_L \frac{w^3}{w+\zeta}\exp(iw^2)\,dw.$$

The term (3.19) can be omitted when Kx is large. For (3.16) shows that $v_0^2\,\Phi_*(\quad)/T_0(K)$ is of the same order as $B_0(\quad)$, and therefore $B_2(\quad)$. Thus the ratio of (3.19) to (3.18) is of order $(Kx)^{-1}F_0\{\quad\}/F_1\{\quad\}$. This is of order $(Kx)^{-1}$, for the ratio $F_3(\zeta)/F_1(\zeta)$ is bounded for all real positive ζ since it tends to finite limits when $\zeta\to 0$ and when $\zeta\to\infty$; also $F_1(\zeta)$ does not vanish when $\zeta \geqslant 0$. (See the appendix for properties of the functions $F_m(\zeta)$.) It follows that

$$\phi_{2K}(x,y,z) \sim \begin{cases} -\dfrac{Kav_0}{2^{\frac{3}{2}}\pi^2}\dfrac{e^{iKx}}{(Kx)^{\frac{1}{2}}}\Phi_*(K,y,z)\,F_1\left\{\left(\dfrac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\} \\ \qquad\qquad \text{near } r=a \text{ when } Kx \gg 1, & (3.20) \\ -\left(\dfrac{1}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{1}{4}i\pi}Ka\dfrac{V_0}{T_0^{(2)}(K)}\dfrac{e^{iKx}}{(Kx)^{\frac{3}{2}}}\Phi_*(K,y,z) \\ \qquad\qquad \text{near } r=a \text{ when } Ka(Kx)^{\frac{1}{2}} \gg 1, & (3.21)\end{cases}$$

on using the asymptotic expansion of $F_1(\zeta)$. Here $v_0(Ka)$ is defined by (4.5) below, and (3.21) now shows that (3.3) is valid not only when Kx is large and Ka is neither large nor small (as shown in IV) but also when Ka is small and Kx is so large that $Ka(Kx)^{\frac{1}{2}}$ is also large.

4. The function H_0 $(Kr,\,\theta,\,Ka)$

We now investigate the function $v_0(Ka)$ in more detail, and show that it is independent of the boundary condition (2.3) on the cylinder. We observe that, if there exists a $\beta_0(Ka)$ such that $S_0(-\beta_0, Ka) = 0$, then the expression (2.14) evidently does not involve $V(k,\theta)$ when $\beta = -\beta_0$. This observation together with (2.15) suggests the following boundary-value problem.

Problem H. Determine a value of β, and coefficients $h_{2m}(\beta, Ka)$ in the expansion

$$H(K\sec\beta,y,z) \equiv \exp(-Kz)\cosh(Ky\tan\beta) - \frac{1}{4\pi}\tan\beta\,\Psi_0(K\sec\beta,y,z)$$
$$+\frac{\cos\beta}{Ka}\sum_{m=1}^{\infty}\frac{h_{2m}(\beta,Ka)}{K'_{2m}(Ka\sec\beta)}\Psi_{2m}(Kr\sec\beta,\theta,\cos\beta), \quad (4.1)$$

such that the boundary condition

$$\langle a\partial H/\partial r\rangle = 0 \quad \text{on} \quad r = a, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi \tag{4.2}$$

is satisfied

We have the following result:

THEOREM H. *For sufficiently small positive Ka the boundary-value problem has no solution unless β takes a certain characteristic value $\beta = \beta_0(Ka)$, where $\beta_0(Ka) \sim 2Ka$ as $Ka \to 0$.* The solution when $\beta = \beta_0(Ka)$ will be denoted by $H_0(Kr, \theta, Ka)$.

It is almost self-evident that the problem has no solution for prescribed Ka and arbitrary β since the boundary-value problems (2.7) and (2.8) above (which have a solution for arbitrary β) contain one more undetermined coefficient than (4.2).

To prove the result, consider the system

$$-\left\langle a\frac{\partial}{\partial r}\left\{e^{-K\delta}\cosh(Ky\tan\beta)-\frac{1}{4\pi}\tan\beta\,\Psi_0\right\}\right\rangle = -\left\langle a\frac{\partial}{\partial r}\left\{\frac{1}{2}\mathscr{T}-\frac{1}{2\pi}\tan\beta\mathscr{R}\right\}\right\rangle$$

$$= \frac{\cos\beta}{Ka}\sum_1^\infty h_{2m}(\beta, Ka)\left\langle\frac{a}{K'_{2m}(ka)}\frac{\partial}{\partial r}\Psi_{2m}\right\rangle + \tfrac{1}{2}h_0(\beta, Ka), \tag{4.3}$$

where $h_0(\beta, Ka)$ is an additional undetermined coefficient. The boundary condition (4.2) in problem H corresponds to the condition $h_0(\beta, Ka) = 0$. We now note that the functions

$$\frac{\cos\beta}{Ka}\left\langle\frac{a}{K'_{2m}(\ \ )}\frac{\partial}{\partial r}\Psi_{2m}\right\rangle$$

tend formally to $\cos 2m\theta$ when $Ka \to 0$, as is readily shown from the known properties of the Bessel functions $K_n(\ \ )$. (See also equation (2.42) of I.) Thus, in the limit, the expansion (4.3) is a Fourier cosine expansion over $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. For general Ka, apply the operators

$$\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \ldots \cos 2n\theta\, d\theta \quad (n = 0, 1, 2, \ldots).$$

The system (4.3) is thus transformed into an infinite system

$$c_{2n} = h_{2n} + \sum_{m=1}^{\infty} a_{nm}h_{2m} \quad (n = 0, 1, 2, \ldots),$$

where c_{2n} is the nth Fourier coefficient of the left-hand side of (4.3), and where all the a_{nm} can be seen to tend to zero when Ka tends to zero. The parameter h_0 does not appear in the equations for $n = 1, 2, 3, \ldots$, and it can be shown that these can be solved for $h_2, h_4, h_6, \ldots$, by iteration when Ka is small enough. It is found that

$$h_{2n} = c_{2n} + O(Ka\max_m|c_{2m}|) \quad \text{when} \quad n = 1, 2, 3, \ldots .$$

Then h_0 can be found from the equation for $n = 0$, i.e. from

$$-\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left\langle a\frac{\partial}{\partial r}\left\{\tfrac{1}{2}\mathscr{T}-\frac{1}{2\pi}\tan\beta\mathscr{R}\right\}\right\rangle d\theta = h_0 + \sum_{m=1}^{\infty} a_{0m}h_{2m}. \tag{4.4}$$

When kr and Ka are small, the expressions for $\mathscr{T}$ and $\mathscr{R}$ in the appendix can be expanded in power series, cf. also § 4 of I. When β is of the same order as Ka, the leading terms on the left-hand side are thus found to be $(2/\pi)\,(Ka - \frac{1}{2}\tan\beta)$ while the terms in the sum on the right are readily seen to be $o(Ka)$, since the h_{2m} are $O(Ka)$ and the a_{0m} are $o(1)$. It follows that $h_0(\beta, Ka) = 0$ for a positive value $\beta = \beta_0(Ka)$ such that $\beta_0(Ka) \sim 2Ka$. Put $\beta = \beta_0(Ka)$ in (4.3); then the coefficients $h_{2m}(\beta_0, Ka)$ can be determined, and the function $H(K\sec\beta_0, y, z)$ is then defined by (4.1). We define $H(K\sec\beta_0, y, z) = H_0(Kr, \theta, Ka)$; this concludes the proof of theorem H.

We note that $H_0(Kr, \theta, Ka)$ satisfies

$$(\partial^2/\partial y^2 + \partial^2/\partial z^2 - K^2\sec^2\beta_0)\,H_0 = 0 \quad \text{in the fluid},$$
$$(K + \partial/\partial z)\,H_0 = 0 \quad \text{on} \quad z = 0, \quad r > a,$$
$$\langle \partial H_0/\partial r\rangle = 0 \quad \text{on} \quad r = a.$$

Also

$$H_0(Kr, \theta, Ka) - \exp(-Kz)\cosh(Ky\tan\beta_0) + (4\pi)^{-1}\tan\beta_0\,\Psi_0(K\sec\beta_0, y, z) = O(\exp(-|Ky|\sec\beta_0)) \quad \text{as} \quad |y| \to \infty.$$

We see that $H_0(\;\;) \to \infty$ exponentially when $|y| \to \infty$ but that the rate of increase $Ky\tan\beta_0$ tends to zero when $\beta_0 \to 0$. We also note that

$$v_0(Ka) = 2\sin\tfrac{1}{2}\beta_0/(\cos\beta_0)^{\frac{1}{2}}, \tag{4.5}$$

where $\beta_0(Ka)$ is defined in theorem H above, and that $v_0(Ka) \sim 2Ka$ when $Ka \to 0$.

5. Problem 1: the action of a fixed long ship on head seas

The treatment in this section is similar to that in § 3 of IV, but some of the approximations made there will not be made here. The diffracted potential is replaced by a potential $\phi_1(x, y, z)$ which satisfies the boundary condition

$$\langle a\,\partial\phi_1/\partial r\rangle = -e^{iKx}h(x)\,\langle a\,\partial e^{-Kz}/\partial r\rangle \quad \text{on} \quad r = a, \tag{5.1}$$

where

$$h(x) = \begin{cases} 0 & \text{on the forward part} \quad -\infty < x \leqslant -l, \\ 1 & \text{on the rear part} \quad\quad l \leqslant x < \infty, \end{cases}$$

and where $h(x)$ is chosen to be an infinitely differentiable increasing function on the middle part $-l \leqslant x \leqslant l$. It is reasonable to hope that this motion will resemble the diffracted waves due to a semi-infinite ship. (The index 1 will usually be omitted.) For the Fourier transform we then obtain the expansion (see equation (3.13) of IV)

$$\Phi(k, y, z) = p_0(k)\,\Psi_0(k, y, z) + \sum_1^\infty p_{2m}(k)\,\frac{\Psi_{2m}(kr, \theta, \cos\beta)}{K'_{2m}(ka)}, \tag{5.2}$$

where

$$\left\langle a\frac{\partial\Phi}{\partial r}\right\rangle = p_0(k)\left\langle a\frac{\partial\Psi_0}{\partial r}\right\rangle + \Sigma\,p_{2m}(k)\left\langle\frac{a}{K'_{2m}(ka)}\frac{\partial\Psi_{2m}}{\partial r}\right\rangle \tag{5.3}$$

$$= -H(k-K)\,\langle\partial e^{-Kz}/\partial r\rangle, \tag{5.4}$$

in which $H(k)$ is the Fourier transform of $h(x)$. Also we have

$$\langle a\,\partial\Psi_0/\partial r\rangle = 2\pi\cot\beta\,T(k, \theta) + R(k, \theta);$$

we note that $\langle a\,\partial e^{-Kz}/\partial r\rangle = T(K, \theta)$. Thus

$$2\pi\cot\beta\,T(k, \theta) + R(k, \theta) = -\frac{H(k-K)}{p_0(k)}\,T(K, \theta) - \Sigma\frac{p_{2m}(k)}{p_0(k)}\left\langle\frac{a}{K'_{2m}(ka)}\frac{\partial\Psi_{2m}}{\partial r}\right\rangle. \tag{5.5}$$

It follows that

$$H(k-K)/p_0(k) = -2\pi\cot\beta\, T_0^{(1)}(k) - 2R_0^{(1)}(k)$$

and

$$p_{2m}(k)/p_0(k) = -2\pi\cot\beta\, T_{2m}^{(1)}(k) - 2R_{2m}^{(1)}(k),$$

whence

$$p_0(k) = -\frac{H(k-K)}{2\pi\cot\beta\, T_0^{(1)} + 2R_0^{(1)}} \tag{5.6}$$

and

$$p_{2m}(k) = -H(k-K)\frac{2\pi\cot\beta\, T_{2m}^{(1)} + 2R_{2m}^{(1)}}{2\pi\cot\beta\, T_0^{(1)} + 2R_0^{(1)}}. \tag{5.7}$$

Here $T_{2m}^{(1)}(k)$ and $R_{2m}^{(1)}(k)$ are coefficients in the expansions

$$T(k,\theta) = T_0^{(1)}(k)\, T(K,\theta) + \Sigma T_{2m}^{(1)}(k)\left\langle \frac{a}{K'_{2m}}\frac{\partial \Psi_{2m}}{\partial r}\right\rangle \tag{5.8}$$

and

$$R(k,\theta) = R_0^{(1)}(k)\, T(K,\theta) + \Sigma R_{2m}^{(1)}(k)\left\langle \frac{a}{K'_{2m}}\frac{\partial \Psi_{2m}}{\partial r}\right\rangle. \tag{5.9}$$

On combining these equations we obtain

$$S(k,\theta) = S_0^{(1)}(\beta, Ka)\, T(K,\theta) + \Sigma S_{2m}^{(1)}(\beta, Ka)\left\langle \frac{a}{K'_{2m}}\frac{\partial \Psi_{2m}}{\partial r}\right\rangle. \tag{5.10}$$

From (5.6) and (5.7) we thus obtain

$$\begin{aligned}\Phi(k,y,z) &= \frac{-H(k-K)}{2\pi\cos\beta\, T_0^{(1)} + 2\sin\beta\, R_0^{(1)}} \\ &\quad\times\left[2\pi\cos\beta\,\mathscr{T} + 2\sin\beta\,\mathscr{R} - \Sigma(2\pi\cos\beta\, T_{2m}^{(1)} + 2\sin\beta\, R_{2m}^{(1)})\frac{\Psi_{2m}}{K'_{2m}}\right] \qquad (5.11)\\ &= -H(k-K)\frac{N^{(1)}(kr,\theta,\beta,Ka)}{S_0^{(1)}(\beta,Ka)} \quad \text{in the notation of (2.12) above.}\end{aligned}$$

Also (see equation (3.20) of IV)

$$H(k-K) = -\frac{i}{k-K} - \int_{-\infty}^{\infty} xh'(x)\,dx + O(k-K),$$

which is seen to be of the form

$$H(k-K) = -\frac{i}{K}\frac{1}{\sec\beta - 1}(1 + \tilde{h}(\beta^2)),$$

where $\tilde{h}(\beta^2) = O(\beta^2)$ when $\beta^2 \to 0$ and where we write $k = K\sec\beta$, as before. Thus

$$\begin{aligned}\phi(x,y,z) &= \frac{1}{2\pi}\int \Phi(k,y,z)\, e^{ikx}\,dk \\ &= \frac{i\,e^{iKx}}{2\pi}\int \frac{(1+\tilde{h}(\beta^2))}{\sec\beta - 1}\frac{\sin\beta}{\cos^2\beta}\exp\left[iKx\,(\sec\beta - 1)\right]\frac{N^{(1)}(kr,\theta,\beta,Ka)}{S_0^{(1)}(\beta,Ka)}\,d\beta \\ &= \frac{i\,e^{iKx}}{2\pi}\int (1+\tilde{h}(\beta^2))\frac{1+\cos\beta}{\sin\beta\cos\beta}\exp\left[iKx(\sec\beta - 1)\right]\frac{N^{(1)}(\quad)}{S_0^{(1)}(\quad)}\,d\beta. \qquad (5.12)\end{aligned}$$

We now observe that near $\beta = 0$ the integrand has two simple poles, one at the saddle point $\beta = 0$, the other at the zero $\beta = -\beta_0(Ka)$ of $S_0^{(1)}(\beta, Ka)$. (The function $\beta_0(Ka)$ is the same as in §4 above, where we noted that it is independent of $V(k,\theta)$.)

As in §3 above we write $v = 2\sin\frac{1}{2}\beta/(\cos\beta)^{\frac{1}{2}}$ but now we expand the non-oscillatory part of the integrand in the form

$$(1+\tilde{h}(\beta^2))\frac{1+\cos\beta}{\sin\beta\cos\beta}\frac{N^{(1)}(\quad)}{S_0^{(1)}(\quad)}\frac{d\beta}{dv} = \frac{C^{(1)}(Kr,\theta,Ka)}{v}+\frac{A^{(1)}(Kr,\theta,Ka)}{v+v_0}+\sum_{m=0}^{\infty}B_m^{(1)}(\quad)v^m, \tag{5.13}$$

to take account of the two simple poles. We require $C^{(1)}(\quad)$, $A^{(1)}(\quad)$ and $B_0^{(1)}(\quad)$. To determine $C^{(1)}$, multiply (5.13) by v and let $\beta\to 0$ and $v\to 0$. Thus

$$2N^{(1)}(Kr,\theta,0,Ka)/S_0^{(1)}(0,Ka) = C^{(1)}(Kr,\theta,Ka).$$

From (5.8) we note that $T_0(K) = 1$ and $T_{2m}(K) = 0$, whence

$$N^{(1)}(Kr,\theta,0,Ka) = 2\pi\mathscr{T} = 2\pi e^{-Kz},\quad S_0^{(1)}(0,Ka) = 2\pi T_0 = 2\pi;$$

thus

$$C^{(1)}(\quad) = 2e^{-Kz}. \tag{5.14}$$

To determine $A^{(1)}(\quad)$, multiply (5.13) by $v+v_0$ and let $\beta\to-\beta_0$ and $v\to -v_0$. Thus

$$A^{(1)}(\quad) = \frac{(1+\tilde{h}(\beta_0^2))\,(1+\cos\beta_0)}{\cos\beta_0(-\sin\beta_0)}\frac{N^{(1)}(Kr\sec\beta_0,\theta,-\beta_0,Ka)}{(\partial S_0/\partial\beta)_{\beta=-\beta_0}},$$

but we shall not make use of this expression. To determine $B_0^{(1)}$ we may apply the integral operator $\int(\ldots)v^{-1}dv$ to both sides, where the contour of integration encloses both the poles. (The calculation is similar to that for equation (3.22) of IV.) We thus find that

$$v_0^{-1}A^{(1)}(\quad)+B_0^{(1)}(\quad) = \pi^{-1}\Phi_*(K,y,z), \tag{5.15}$$

where $B_0^{(1)}(\quad)$ is smaller than the other two terms by a factor $O(Ka)$. The result (5.15) can also be obtained from equation (3.25) of IV by considering the very distant field where $Ka(Kx)^{\frac{1}{2}}\gg 1$; cf. (3.16) above. Substituting (5.13) in (5.12), and proceeding as in §3 above, we find that

$$\begin{aligned}\phi_{1K}(x,y,z) \sim{}& \frac{i\,e^{iKx}}{2\pi}C^{(1)}(Kr,\theta,Ka)\oint_L \exp(\tfrac{1}{2}iKxv^2)\frac{dv}{v}\\ &+\frac{i\,e^{iKx}}{2\pi}A^{(1)}(Kr,\theta,Ka)\int_L\frac{\exp(\frac{1}{2}iKxv^2)}{v+v_0}dv\\ &+\frac{i\,e^{iKx}}{2\pi}\sum_{m=0}^{\infty}B_{2m}^{(1)}(\quad)\int_L v^{2m}\exp(\tfrac{1}{2}iKxv^2)\,dv,\end{aligned} \tag{5.16}$$

where in the first term the path of integration passes below $v = 0$; thus the integral in this term has the value πi since the integrand is an odd function of v along L. Using (5.14) we find that

$$\begin{aligned}\phi_{1K}(x,y,z) \sim{}& -e^{-Kz}e^{iKx}+\frac{i\,e^{iKx}}{2\pi}A^{(1)}(\quad)\int\frac{\exp(\frac{1}{2}iKxv^2)}{v+v_0}dv\\ &+\frac{i\,e^{iKx}}{2\pi}B_0^{(1)}(\quad)\int\exp(\tfrac{1}{2}iKxv^2)\,dv\\ ={}& -e^{-Kz}e^{iKx}+\frac{iv_0}{2\pi^2}\Phi_*(K,y,z)\int\frac{\exp(\frac{1}{2}iKxv^2)}{v+v_0}dv+\frac{i\,e^{iKx}}{2\pi}B_0^{(1)}(\quad)\\ &\times\int\frac{v\exp(\frac{1}{2}iKxv^2)}{v+v_0}dv\end{aligned} \tag{5.17}$$

on using (5.15). This represents the diffracted wave; the total wave field for large x is obtained by adding the incident wave $e^{-Kz}e^{iKx}$. Thus

$$\phi_{\text{inc}}+\phi_{1K} \sim \frac{iv_0(Ka)}{2\pi^2}\, e^{iKx}\,\Phi_*(K,y,z)\,F_0\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0\,(Ka)\right\} \tag{5.18}$$

$$+\frac{i\,e^{iKx}}{\pi(2Kx)^{\frac{1}{2}}}\,B_0^{(1)}(Kr,\theta,Ka)\,F_1\left\{\left(\frac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\} \tag{5.19}$$

$$+\,e^{iKx}O((Kx)^{-\frac{3}{2}}),$$

where

$$F_m(\zeta) = \int_L \frac{w^m}{w+\zeta}\exp\,(iw^2)\,dw, \quad m = 0, 1. \tag{5.20}$$

As in § 3 above, we can now show that for large Kx the term (5.19) is of smaller order of magnitude than (5.18). For it can be seen that $v_0\,\Phi_*(\quad)$ and $B_0(\quad)$ are both of order unity; thus the ratio of (5.19) to (5.18) is of order $(Kx)^{-\frac{1}{2}}F_1\{\quad\}/F_0\{\quad\}$, and the ratio $F_1(\zeta)/F_0(\zeta)$ can be shown to be bounded for all positive ζ. For this is certainly the case when $\zeta\to 0$ and when $\zeta\to+\infty$; also $F_0(\zeta)$ does not vanish for positive real ζ. (See appendix for these properties.) The boundedness of the ratio $F_1(\zeta)/F_0(\zeta)$ follows. Thus (5.19) may be omitted, and

$$\phi_{\text{inc}}+\phi_{1K} \sim \begin{cases} \dfrac{iv_0(Ka)}{2\pi^2}\, e^{iKx}\,\Phi_*(K,y,z)\,F_0\left\{\left(\dfrac{Kx}{2}\right)^{\frac{1}{2}} v_0\,(Ka)\right\} \\ \qquad\qquad \text{near} \quad r=a \quad \text{when} \quad Kx \gg 1, & (5.21) \\ -\dfrac{1}{\pi}\, e^{-\frac{1}{4}i\pi}\left(\dfrac{1}{2\pi Kx}\right)^{\frac{1}{2}} e^{iKx}\,\Phi_*(K,y,z) \\ \qquad\qquad \text{near} \quad r=a \quad \text{when} \quad Ka(Kx)^{\frac{1}{2}} \gg 1. & (5.22) \end{cases}$$

Here $v_0(Ka)$ is defined by (4.5) above. The result (5.22) was obtained in IV under the more restrictive condition that Kx is large and Ka is neither large nor small.

6. Comparison with the thin-ship calculation

As was mentioned in § 1 above, the calculation given in § 5 of II can be interpreted as the diffraction of an incident head sea by a thin ship. The normal velocity on the plane $y = 0$ was assumed to vanish when $x < 0$ and was assumed to be proportional to $e^{-Kz}e^{iKx}$ when $x > 0$. It was then found that the resulting (diffracted) wave ultimately increases like $(Kx)^{\frac{1}{2}}$ when $x\to\infty$ along the plane, whereas the variation along a ship was found in IV to decrease ultimately like $(Kx)^{-\frac{1}{2}}$. These results appear to be inconsistent, but we shall now show that the apparent inconsistency can be removed by an appropriate interpretation.

Let us first see how the calculation of § 5 of II can be interpreted as the diffraction of a head sea by a thin ship. For this purpose let us consider a semi-infinite ship of thin triangular cross-section, draught d comparable to $2\pi/K$, small semi-vertical angle ϵ and small beam $2d\tan\epsilon$. The normal velocity induced by the incident wave $e^{-Kz}e^{iKx}$ on this wedge is (cf. (5.1) above)

$$e^{iKx}(-\partial e^{-Kz}/\partial n) \doteqdot -K\sin\epsilon\, e^{iKx}e^{-Kz} = -(Kb/d)\,e^{iKx}e^{-Kz}$$

when $0 < x < \infty$ and $0 < z < d$; at all other points of the mid-plane $y = 0$ the normal velocity vanishes. This is nearly the same velocity distribution as in §5 of II, but with a multiplying factor b/d, and it can be shown that it gives rise to nearly the same diffracted wave near $y = 0$, which with the same factor is thus

$$O((b/d)(Kx)^{\frac{1}{2}}) = O(Kb(Kx)^{\frac{1}{2}}).$$

Thus the total wave near the thin ship is $1 + O(Kb(Kx)^{\frac{1}{2}})$ when Kx is large and Kb is very small.

Let us next consider the amplitude variation near the semicircular ship of §5 above when Ka is small, Kx is large and $Ka(Kx)^{\frac{1}{2}}$ is small. From (5.21) this is given by

$$F_0\{\tfrac{1}{2}Kx)^{\frac{1}{2}} v_0(Ka)\} = \pi i - 2i\pi^{\frac{1}{2}} e^{\frac{1}{4}i\pi}(\tfrac{1}{2}Kx)^{\frac{1}{2}} v_0(Ka) + \ldots$$
$$\doteqdot \pi i\{1 - (8\pi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} Ka(Kx)^{\frac{1}{2}} + \ldots\},$$

which (except for constant factors) is of the same form as for the thin ship. We can now suggest the following interpretation for the $(Kx)^{\frac{1}{2}}$ variation found for the thin ship in §5 of II: the term $(Kx)^{\frac{1}{2}}$ should have been multiplied by a factor of order Kb, where $2b$ is the beam; the resulting total-amplitude variation $1 + O(Kb(Kx)^{\frac{1}{2}})$ is then valid when Kx is large but $Kb(Kx)^{\frac{1}{2}}$ is small. This interpretation is consistent with §5 of II, since it was implicitly assumed there that Kb was infinitely small.

7. Summary of results and discussion

We have considered the two problems described in §1 which were previously treated in IV, but have assumed here that the dimensionless wavenumber Ka may be small. Problem 1 is concerned with head seas incident on a fixed semi-infinite cylindrical ship of semicircular cross-section; it is found that the total wave potential is given (asymptotically for large Kx) by

$$\phi_{\text{inc}} + \phi_{1K} \sim \begin{cases} \dfrac{iv_0(Ka)}{2\pi^2} e^{iKx}\, \Phi_*(K,y,z)\, F_0\left\{\left(\dfrac{Kx}{2}\right)^{\frac{1}{2}} v_0(Ka)\right\} \\ \qquad\qquad \text{near} \quad r = a \quad \text{when} \quad Kx \gg 1, & (7.1) \\ -\dfrac{1}{\pi} e^{-\frac{1}{4}i\pi}\left(\dfrac{1}{2\pi Kx}\right)^{\frac{1}{2}} e^{iKx}\Phi_*(K,y,z) \\ \qquad\qquad \text{near} \quad r = a \quad \text{when} \quad Ka(Kx)^{\frac{1}{2}} \gg 1. & (7.2) \end{cases}$$

Here the function $\Phi_*(K, y, z)$ is the two-dimensional potential defined in §3 above; the function $v_0(Ka)$ is defined by the eigenvalue problem described in §4 above and satisfies $v_0(Ka) \sim 2Ka$ as $Ka \to 0$; and the function $F_0(\zeta)$, defined in the appendix, is related to the Fresnel integral as would be expected in a problem of glancing incidence. The argument $(\frac{1}{2}Kx)^{\frac{1}{2}} v_0(Ka)$ (which may be small, intermediate or large) is the product of a large factor $(\frac{1}{2}Kx)^{\frac{1}{2}}$ and a small factor $v_0(Ka)$, and this also would be expected in such a problem. [Cf., for instance, the expression for the pressure on a semi-infinite plane due to a plane sound wave at nearly glancing incidence. This is given in many textbooks, e.g. in Jones (1964, p. 588).] When $Ka(Kx)^{\frac{1}{2}}$ is large, the asymptotic expansion of $F_0(\zeta)$ is applicable and leads to (7.2). This is the same expression as that in equation (3.26) of IV, which is thus seen to be valid not only when Ka is of order unity and Kx is large but also when Ka is small and $Ka(Kx)^{\frac{1}{2}}$ is large. The decay factor $(Kx)^{-\frac{1}{2}}$ in (7.2) has no analogue in the acoustic problem.

Problem 2 is concerned with waves generated by a pulsating source on a fixed infinite ship of semicircular cross-section. It is found that the wave potential for large Kx is given asymptotically by

$$\phi_{2K} \sim \begin{cases} -\dfrac{Kav_0^2}{2^{\frac{3}{2}}\pi^2}\dfrac{V_0}{T_0^{(2)}(K)}\dfrac{e^{iKx}}{(Kx)^{\frac{1}{2}}}\Phi_*(K,y,z)\,F_1\left\{\left(\dfrac{Kx}{2}\right)^{\frac{1}{2}}v_0(Ka)\right\} \\ \qquad\qquad\qquad \text{near} \quad r=a \quad \text{when} \quad Kx \gg 1, & (7.3) \\ -\left(\dfrac{1}{2\pi}\right)^{\frac{1}{2}}e^{-\frac{1}{4}i\pi}\dfrac{V_0}{T_0^{(2)}}\dfrac{e^{iKx}}{(Kx)^{\frac{3}{2}}}\Phi_*(K,y,z) \\ \qquad\qquad\qquad \text{near} \quad r=a \quad \text{when} \quad Ka(Kx)^{\frac{1}{2}} \gg 1. & (7.4) \end{cases}$$

The function $F_1(\zeta)$ is also defined in the appendix, and the remarks which have just been made about problem 1 are equally applicable to problem 2.

The calculation in § 5 of II showed that for a semi-infinite thin ship the diffracted wave increases like $(Kx)^{\frac{1}{2}}$. The re-examination in § 6 above shows that this must be multiplied by a scaling factor of order Kb, where $2b$ is the (infinitely small) beam of the thin ship. Thus the diffracted wave is in fact of order $Kb(Kx)^{\frac{1}{2}}$, and comparison with problem 1 suggests that this is valid when Kx is large but $Kb(Kx)^{\frac{1}{2}}$ is small.

Equation (7.1) shows that the refraction away from the cylinder becomes significant when the product $Ka(Kx)^{\frac{1}{2}}$ becomes large, i.e. when $Kx \gg (Ka)^{-2}$. The function $v_0(Ka)$ which appears in this calculation is determined by the eigenvalue problem of § 4 above. Its solution involves analytic continuation to the second (non-physical) sheet of the k plane, or to the non-physical part of the β plane, and it is thus difficult to give a physical interpretation of the calculation of $v_0(Ka)$.

We note once again that the calculations of the present paper use little more than the analytic form of the Fourier transform near $k = K$, and that this can be found fairly easily from the form of the expansion (2.6), which however is applicable only to the semicircle. For other cross-sections it will be necessary to rephrase the argument in terms of integral equations, and this will also be necessary for the study of the other limiting case when Ka is large, about which little is known.

Appendix

The functions $\mathscr{T}$, $\mathscr{R}$, Ψ_{2m}

These are defined in § 2 of IV (here we have used the relation $k = K\sec\beta$):

$$\mathscr{T}(kr,\theta,\cos\beta) = \exp(-kz\cos\beta)\cosh(ky\sin\beta),$$

$$\mathscr{R}(kr,\theta,\cos\beta) = -\beta\cot\beta\exp(-kz\cos\beta)\cosh(ky\sin\beta) + K_0(kr+2)\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{\partial}{\partial\nu}(I_\nu(kr)\cos\nu\theta)\right]_{\nu=m}\sin m\beta\cot\beta,$$

$$\Psi_0(kr,\theta,\cos\beta) = 2\pi\cot\beta\,\mathscr{T}(kr,\theta,\cos\beta) + 2\mathscr{R}(kr,\theta,\cos\beta) \quad \text{when} \quad k > K;$$

when $k < K$, put $\beta = -i\alpha$, where $\alpha > 0$, so that

$$\Psi_{2m}(kr,\theta,\cos\beta) = K_{2m}(kr)\cos 2m\theta + 2\cos\beta\,K_{2m-1}(kr)\cos(2m-1)\theta + K_{2m-2}(kr)\cos(2m-2)\theta, \quad m = 1,2,3,\ldots.$$

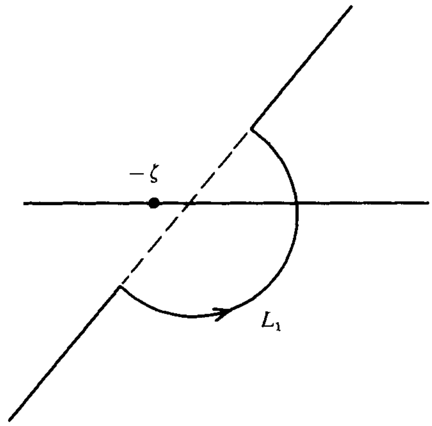

FIGURE 1

Here the functions $I_\nu(\zeta)$ and $K_\nu(\zeta)$ are Bessel functions in the usual notation. For the expansions of these potentials for small kr, see §§ 2 and 4 of I.

Expansion of the functions $F_m(\zeta)$ for small and large positive ζ

By definition we have

$$F_m(\zeta) = \int_L \frac{w^m}{w+\zeta} \exp(iw^2)\,dw,$$

where the path of integration L goes from $w = -\infty \exp(\frac{1}{4}i\pi)$ through $w = 0$ to $w = \infty \exp(\frac{1}{4}i\pi)$, and where $-\zeta < 0$ lies to the left of L. To obtain the expansion for small ζ, deform the path of integration into the path L_1, which near $w = 0$ has the form of the semicircle $|w| = l$, $-\frac{3}{4}\pi < \arg w < \frac{1}{4}\pi$, where $l > |\zeta|$ (see figure 1).

On L_1 we have $|w| > |\zeta|$, whence

$$F_m(\zeta) = \int_L \frac{w^m}{w+\zeta} \exp(iw^2)\,dw = \sum_{k=0}^{\infty} (-\zeta)^k \int_{L_1} w^{m-k-1} \exp(iw^2)\,dw = \sum_{k=0}^{\infty} (-\zeta)^k q_{k+1-m}, \text{ say.}$$

By integration by parts we have

$$2iq_{s-1} = sq_{s+1}. \tag{A 1}$$

Also evidently

$$q_0 = \int_{L_1} \exp(iw^2)\,dw = \pi^{\frac{1}{2}} e^{\frac{1}{4}i\pi}; \tag{A 2}$$

and

$$q_1 = \int_{L_1} w^{-1} \exp(iw^2)\,dw = \pi i, \tag{A 3}$$

for the integrand is odd and the straight parts of L_1 therefore give a total contribution of zero. It follows that $q_s = \pi \exp(\frac{1}{4}is\pi + \frac{1}{4}i\pi)/\Gamma(\frac{1}{2}s + \frac{1}{2})$, whence we have the convergent expansion

$$F_m(\zeta) = i\pi \exp(-\tfrac{1}{4}im\pi) \sum_{k=0}^{\infty} \frac{(-\zeta e^{\frac{1}{4}i\pi})^k}{\Gamma(1+\frac{1}{2}k-\frac{1}{2}m)}. \tag{A 4}$$

We are concerned with the functions F_0, F_1 and F_3, and we note that none of these vanishes when $\zeta = 0$.

When $\zeta \gg 1$ we use Watson's lemma:

$$F_m(\zeta) \sim \int w^m \left(\frac{1}{\zeta} - \frac{w^2}{\zeta^2} + \ldots\right) \exp(iw^2)\, dw$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{\zeta^{k+1}} \int w^{m+k} \exp(iw^2)\, dw = \sum_{k=0}^{\infty} \frac{(-1)^k}{\zeta^{k+1}} q_{-m-k}.$$

In particular, $F_0(\zeta) \sim \pi^{\frac{1}{2}} e^{\frac{1}{4}i\pi} \zeta^{-1}$, $F_1(\zeta) \sim \frac{1}{2}\pi^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}\zeta^{-1}$ and $F_3(\zeta) \sim \frac{3}{4}\pi^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\zeta^{-2}$, since $q_{-1} = q_{-3} = \ldots = 0$. Here (A 1) and (A 2) have been used. It follows that the ratios F_3/F_1 and F_1/F_0 are bounded for large positive ζ.

It remains to show that $F_0(\zeta)$ and $F_1(\zeta)$ do not vanish for positive ζ. We have

$$F_0(\zeta) = \int_{-\infty}^{\infty} \frac{e^{\frac{1}{4}i\pi} \exp(-\rho^2)\, d\rho}{\zeta + \rho\, e^{\frac{1}{4}i\pi}} = \int_{-\infty}^{\infty} \frac{e^{\frac{1}{4}i\pi} \exp(-\rho^2)\, d\rho(\zeta + \rho\, e^{-\frac{1}{4}i\pi})}{(\zeta + \rho\, e^{\frac{1}{4}i\pi})(\zeta + \rho\, e^{-\frac{1}{4}i\pi})}.$$

Thus the imaginary part of $F_0(\zeta)$ is

$$\zeta \sin \tfrac{1}{4}\pi \int_{-\infty}^{\infty} \frac{\exp(-\rho^2)\, d\rho}{|\zeta + \rho\, e^{\frac{1}{4}i\pi}|^2},$$

which clearly does not vanish when ζ is real, except possibly when $\zeta = 0$. Since $F_0(0) \neq 0$, it follows that $F_0(\zeta)$ does not vanish for real positive ζ.

Similarly we have

$$F_1(\zeta) = \int_{-\infty}^{\infty} \frac{i\pi \exp(-\rho^2)\, d\rho}{\zeta + \rho\, e^{\frac{1}{4}i\pi}} = \int_{-\infty}^{\infty} \frac{i\rho(\zeta + \rho\, e^{-\frac{1}{4}i\pi}) \exp(-\rho^2)\, d\rho}{|\zeta + \rho\, e^{\frac{1}{4}i\pi}|^2}.$$

Thus the real part of $F_1(\zeta)$ is

$$\cos \tfrac{1}{4}\pi \int_{-\infty}^{\infty} \frac{\rho^2 \exp(-\rho^2)\, d\rho}{|\zeta + \rho\, e^{\frac{1}{4}i\pi}|^2},$$

which clearly does not vanish; so $F_1(\zeta)$ does not vanish for real positive ζ.

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